USING
THE CRANMER ABACUS
FOR THE BLIND

By

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BACKGROUND

The Cranmer Abacus is an abacus which has been modified for the blind. For centuries, man has made use of the abacus as a computer of arithmetic, and, in fact, modern digital computers are based on this ancient reckoning board. The Romans made widespread use of the abacus and modern scholarship believes that this device, generally thought to have originated in the Orient, actually was introduced into the Orient through trade with ancient Rome.

In addition to its popularity in the Orient, the abacus enjoyed considerable use in Europe until the sixteenth or seventeenth centuries. At about this time, writing materials such as pen, ink, and paper became fairly plentiful and abacus calculation began to fall into disuse, giving place to graphic or written arithmetic.

Meanwhile, in the Orient, instead of faltering, the abacus flourished, receiving widespread attention from Chinese and Japanese mathematicians and other scholars. Today, its popularity in Japan alone is enough to stagger the imagination. To cite just one example of its popularity, ninety (90) per cent of all business arithmetic in Japan is done on the abacus, or soroban as it is called in Japanese. There are many reasons for the popularity of the abacus. *Speed:* The speed of abacus calculation is nothing short of breath-taking to someone witnessing an abacus demonstration for the first time. In contest after contest, skilled sighted abacus operators have outperformed skilled sighted operators of electric calculators, both in speed and in accuracy. *Economy:* The abacus is inexpensive to buy and costs nothing to maintain. If writing materials are in short supply, another saving is realized in that only the final answer of a difficult set of calculation need be recorded, thus doing away with the need of pages and pages of so-called "scratch paper". *Portability:* The abacus is small, lightweight, and easy to carry. Unlike modern calculating equipment, which requires a source of electrical power or at least replacement of paper tape, the abacus is self-contained and, if you have the device, you have everything needed to do the most complicated of calculations. *Freedom from Mental Effort:* Another compelling reason for the popularity of the abacus is the fact that, once having mastered the process of abacus calculations, the operator is completely free from the need to do mental arithmetic, for abacus calculation is completely automatic and mechanical in nature.

Unfortunately, the abacus, as used by sighted people, does not lend itself to operation by the blind. The incredible speed of the device stems from the fact that the counters are free to travel back and forth on their columns with virtually no resistance. The merest touch of a finger will move a counter. This condition, which makes the abacus so useful to sighted people, makes it worthless to a blind operator.

DEVELOPMENT OF THE CRANMER ABACUS

T. V. Cranmer, Director of the Division of Services for the Blind, Kentucky Department of Education, is extremely interested in radio and electronics. One day, while working with a set of extremely complex electronic formulas, Cranmer conceived the idea of using the abacus to do the difficult calculations. As we have already seen, the unmodified abacus is of little or no value to the blind. So, it was not long before Cranmer was busily at work designing and building an abacus which could be read by touch. Having a useable abacus was only half the battle. Next came the problem of learning to operate the device. Attainment of this goal would have been impossible were it not for the man to whom this book is dedicated, Mr. Takashi Kojima, of Tokyo, Japan. Through the pages of the book, THE JAPANESE ABACUS, ITS USE AND THEORY, published by the Charles E. Tuttle Company of Rutland, Vermont, Mr. Kojima became our teacher and friend. Through his book, he has given us the most priceless gift one man can bestow upon another, the gift of knowledge. It is for this reason that this work is dedicated to him.

THE PROMISE OF THE ABACUS

Soon, Cranmer and those of us associated with him in this work, began exploring the mysteries of abacus calculations. The deeper we delved the more we learned. The more we learned, the more fascinated we became. For the first time in our lives, we as blind people were able to do extremely complex arithmetic computations quickly, efficiently, effortlessly, on a pocket-sized, one-piece device. We could add, subtract, multiply, divide, extract roots, handle decimals, fractions, and, in short, do all sorts of elaborate calculations in but a small fraction of the time heretofore required.

Those of us who saw the Cranmer Abacus born, and who watched it grow from the beginning, are thrilled by what we like to call "the promise of the abacus", that is, by its potential worth as an aid to our fellow blind people: to the student, it offers greater insight into the processes of arithmetic; to the blind man in business, it offers a considerable saving in time in the performance of necessary business calculation; to the housewife, the lightning-fast method of keeping tabs on the family's bank account; to the blind hobbyist, interested in science and mathematics, the ability to handle quickly, efficiently, and with pinpoint accuracy, the most difficult of arithmetic computations. To all blind people, the abacus promises to remove from arithmetic calculations the awkwardness and drudgery which has served so long to discourage blind people from its study and enjoyment, and to make arithmetic an area into which a blind person can walk with confidence and competence.

HOW TO STUDY

This text is an instruction manual intended to teach you just how to go about using the Cranmer Abacus for addition, subtraction, multiplication, division, the extraction of roots, treatment of fractions, and decimals. Therefore, we are assuming that, as you read this text, you, the student, will have at hand a working model of the abacus and will follow along with your abacus the instructions given here. Once you have gained proficiency in the use of the abacus, you should be able to make arithmetic calculations far more quickly than a sighted person using a pencil and paper. While we do not expect blind people to attain the speed of a sighted abacus operator, the speed which we blind people are able to attain, coupled with the ease and comfort of calculation, provides ample reward for the serious student.

In learning to use the abacus, just as in developing any other set of skills, the key to success is practice. It will not be enough for you to read or listen to this text once or twice. Instead, you should begin at the beginning practicing with your abacus every step of the way. You should not take on new material until you have learned that which you have already read. Trying to go too far too fast will be like building a house upon a poor foundation. A good motto for you to keep in mind would be, "take your time, but learn it well".

MEET THE ABACUS

The abacus is small enough to fit into a man's pocket or a woman's purse. Its frame is oblong in shape. Anchored to this frame are thirteen rods or columns. On each column there are five beads which travel back and forth. Running the entire length of the frame, so that it cuts across all thirteen columns, is the separation bar. From now on we shall call this simply the bar. On every column, four of the five beads or counters are separated from the remaining bead by this bar.

In order to show you the correct operating position for the abacus, imagine the abacus to be lying upon a page in a Braille book. In this position the length of the abacus would be parallel to the lines of Braille, with the columns cutting across the lines or running from the top of the page to the bottom. In the operating position, the bar which separates one bead from its four companions on each column is in the upper portion of the abacus, so that the single beads are close to the upper member of the frame.

If you run your finger along the face of the lower edge of the frame you will feel a dot at the end of each column. In addition, you will feel a line between columns and these lines will be spaced three columns apart. The dots are to help you to locate your place most easily, and the lines, called *unit marks*, serve as decimals.

Instructions given here assume that the abacus operator is right-handed. Unfortunately, left-handed people seem to have to go through life constantly modifying their behavior to meet the demands of a "right-handed world". Learning to operate the Cranmer Abacus should present no serious problem.

THE LANGUAGE OF THE ABACUS

When we enter a number of our abacus we do not say we write it. Instead, we use the word "set". To remove or erase a number we say that we "clear it". In telling you how to use the abacus, we will use such expressions as, "set 5", or "clear 8", or "clear 8, set 1 left". This means clear 8 and set 1 on the column immediately to the left. Or, we might say, "clear 1 left, set 3". This means remove 1 from the column immediately to the left and set 3 on the column on which we are working.

SETTING NUMBERS

The beads or counters which are used to represent numbers take on their value when they are moved close to the bar. They lose their value when they are moved away from the bar. The correct use of the counters on a single column makes it possible to show the ten digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, on a single column. Only one digit can be shown on a column at any given time. To set a four-digit number requires the use of four columns. An eight-digit number takes eight columns. And a one-digit number just one column. The number 365 is made up of the three digits, 3, 6, 5. Therefore, to set it we must use three columns. When all of the beads on a column are moved away from the bar, that column either represents the digit 0 or has no value because it is unused in a given problem. Usually you will have no trouble deciding which is the case.

We have already stressed the importance of daily practice and it has been for this reason that the abacus was designed with only thirteen columns. If it were larger it could not be carried in pocket or purse, within easy reach for short practice sessions. However, there may be times when you will need more than thirteen columns. In such cases, simply join two or more abacuse end to end. Every time you add another abacus you will have available to you thirteen additional columns.

Each of the four beads closest to you on a column has a value of 1. The single bead above the bar has a value of 5. To set the number 1, we would move one of the lower beads on a column up to the bar. It would, of course, have to be the bead already closest to the bar. To set the number 2, you would move two lower beads up to the bar. To set 3, move three lower beads to the bar. To set 4, move all four lower counters up to the bar. To set the number 5, we would move an upper bead down to the bar, on a given column, but the four beads previously set would have to be cleared by moving them as far away from the bar as possible. Only the one upper bead would be close to the bar. To set the number 6, we would move down one upper bead and move up one lower bead upon the same column. Five plus 1 equals 6. To set 7, set 5 and on the same column set 2. Seven is 5 plus 2. To set 8, set 5 above the bar and 3 below. To set 9, all of the beads on a column are moved as close to the bar as possible. That is, the upper bead is moved down and all four lower beads are moved up.

It is very important that you practice the setting of numbers since you cannot hope to use the abacus until you are able to do this. For practice, set such numbers as the year, your telephone number, your street number, the telephone or street number of a friend, the numbers one at a time from 1 to 100, etc. Once you find that you are able to set and read numbers quickly and accurately, you are ready to begin learning how to add with the abacus.

Of the four processes of arithmetic—addition, multiplication, subtraction, division—addition is the one most often used. The operations which we will learn and use in addition are also very important to multiplication, subtraction, and division. Therefore, it is *most* important that we learn how to add and how to add well before we learn anything else.

ADDING ONE-DIGIT NUMBERS

We begin learning to add by adding the number 1 to itself a number of times. We will set the number 1 and continue to add it to the right-most column of the abacus. In order that we will all be talking the same language, we will label this column column c. The one immediately to its left will be column b and the one immediately to the left of column b will be column a. All the remaining columns on the abacus will be unused in our work for the present time. This labeling of columns with letters will only serve to identify columns in so far as they play a part in a given problem. The labeling is arbitrary, and where, in one case, column a may be near the right end of your abacus, in another it may be at the middle or left. On column c set the number 1 by moving one of the lower beads up to the bar. Next, add 1 to this 1. Again add another 1. Now, add still one more 1. Column c should now be showing the number 4. If we try to add one more we find that we cannot add it directly, that is, we cannot move a lower bead up to the bar because there are no more lower beads to move up. All of them have been moved up to the bar to indicate the number 4. So, in order to add 1 to this 4, we move the index finger of our right hand across the bar and up to the top of the column. There, in one continuous motion, we slide the 5 bead down to the bar, move our finger down across the bar, and slide the four lower beads away from the bar and down to the bottom of the column. By doing this, we add 1 to 4 and now have the digit 5 showing on column c. In order to add 1 to this 5 we do as we did before. We move one lower counter up to the bar, so that above the bar we have our 5, and immediately below it is our 1 counter. Five plus 1 equals 6. Add 1 to the 6 and you will have 7. Add 1 to the 7 and you will have 8. Add 1 to the 8 and column c will show the digit 9. And now, because on column c all the counters have been moved against the bar, we again find ourselves unable to add the digit 1 directly. The second rule, or secret, for adding 1 indirectly is stated: to add 1, clear 9, set 1 left. In other words to apply this rule we must remove the 9 from column c and set 1 on column b. Column b now shows 1, column c 0, and the number 10, that is the digits 1 0, appear on columns b and c. If we want to add still more ones then we must set them on column c. To the 10 on columns b c add 1 by setting 1 on column c. We now have the digits 1 1 on columns b and c. This represents the number 11. Now, continue adding ones as before. When you get to 14 you will again have to use the first rule for the indirect addition of 1, that is, set 5, clear 4 with the downward sweep of the index finger. This will give you a total of 15. Again, when you have reached 19 you will have to apply the second rule or secret for the indirect addition of 1 which says: to add 1, clear 9, set 1 left. This will give you 2 on b, 0 on c, and the number 20 on columns b c. Now on column c add 1. This gives 21. Continue until you reach 44. Again, set 5, clear 4 with the downward sweep of the index finger on column c, and this will give 45. Continue adding 1 to the 45 until you reach the number 49. Once again it is necessary to clear 9 and set 1 left, that is, clear the 9 from c and set 1 on column b. In order to set 1 on column b you will have to to do it indirectly. That is, you will have to set 5 and clear 4 on column b with another downward sweep of the index finger; only in this case, instead of setting 5, clearing 4, you will be setting 50, and clearing 40. Now continue to add 1 on up through the 50's, 60's, 70's, 80's and 90's. Eventually, you will reach 98. Add 1 and you will have 99, and it again becomes necessary to add 1 indirectly. To do this, clear 9, set 1 left. Clear the 9 from column c. Then, notice that it is impossible to set 1 on column b because all of the counters on b are active, that is, are against the bar. So, you must clear 9 from b and set 1 on column a. This gives you 1 on a and 0's on b and c for the three-digit number 100.

THE SECRETS OF ADDITION

We have already told you that the Cranmer Abacus is based upon the Japanese Abacus. In using the abacus, when it is impossible to add the number directly, such as we have already seen in the case of 4 plus 1, 9 plus 1, the Japanese make use of indirect means. The number 5 can only be added in two ways, one direct and one indirect. The Japanese have called the indirect methods of adding numbers *secrets*, and Japanese school children, learning to use the soroban, memorize these secrets until their use becomes automatic.

Soon we will list all the secrets for addition in case you would like to memorize them. But first, for those who would rather learn the basic principle upon which the abacus operates, we will try to explain the basis for use of the secrets. We must make use of one of the secrets for addition when it is not possible for us to add a number directly. For example, 4 plus 1 equals 5. Set the number 4 on a column. You will note that with all 4 one-unit counters moved up to the bar, it is not possible to add 1 directly. Four plus 1 is the same as 4 plus (5 minus 4).

At first this may seem quite complicated, but think of it this way. Suppose in your pocket you have 4 pennies. Suppose, too, that I owe you a penny and am willing to pay you. Unfortunately, I do not have a single penny, but I do have a five-cent piece. If I give you the five-cent piece and you, in turn, give me four cents change, what I have actually done is to add one cent to the four cents which you already had, giving you a grand total of five cents. To put it into the language of the rule or secret, into your pocket I have set 5, cleared 4 cents.

The same thing is true when we consider the problem 9 plus 1 equals 10. Nine plus 1 is the same as 9 minus (9 plus 10). What was true of the five-cent piece is also true of the ten-cent piece. If you have nine cents and I want to give you a penny but have no one-cent piece, I can give you a dime, and you in turn can give me your nine cents. This results in the addition of 1 cent to 9, for a total of 10 cents.

The secrets make use of what are called *complementary numbers*. These might also be called *difference numbers*. These are numbers that represent the difference between the number you want to add and either 5 or 10. The complementary numbers for 10, are 1 and 9, 2 and 8, 3 and 7, 4 and 6, and 5 and 5. If you would gain insight into the use of the complementary numbers it should help you to keep in mind the principles of making change in a transaction involving the exchange of money. To give you another example of the application of a secret, consider 8 plus 7 equals 15. We have 8 set on a column, and to this 8 we want to add 7. The difference or complementary number for 7, in order to arrive at 10, is 3. Eight plus 7, then, is the same as saying 8 minus (3 plus 10). Eight plus 7 equals 8 minus 3, which is 5 plus 10, for a total of 15. Let's do this same example in terms of money. Eight cents plus seven cents equals fifteen cents. Now, eight cents plus seven cents is the same as eight cents minus three cents, which is five cents, plus ten cents, for a total of fifteen cents. Suppose I have eight cents, and you owe me seven cents. Unfortunately. you do not have seven cents, but you do have a dime. By applying this secret of addition you will be able to pay me the seven cents. If I give you three cents I will have five cents left; eight minus three is five. If you in turn give me your dime, we will have added ten cents to this five cents for a total of fifteen cents. So, as we have already said, 8 plus 7 is the same as 8 minus (3 plus 10).

The secrets for addition are as follows:

TO ADD	SECRET
1	Set 5, clear 4
1	Clear 9, set 1 left
2	Set 5, clear 3
2	Clear 8, set 1 left
3	Set 5, clear 2
3	Clear 7, set 1 left
4	Set 5, clear 1
4	Clear 6, set 1 left
5	Clear 5, set 1 left
6	Set 1, clear 5, set 1 left
6	Clear 4, set 1 left
7	Set 2, clear 5, set 1 left
7	Clear 3, set 1 left
8	Set 3, clear 5, set 1 left
8	Clear 2, set 1 left
9	Set 4, clear 5, set 1 left
9	Clear 1, set 1 left

Once again, to set 1 left means to set the digit 1 on the column immediately to the left of the one on which we are working. If this column contains a 9 you of course cannot set 1 on it directly. So, clear the 9, and set 1 on the next column over to the left.

PRACTICE EXERCISES

In order for you to gain skill in the use of the abacus it will be necessary that you practice, practice, and practice again. An exercise which will help you to gain skill is to begin with the number 1; to this, add 2 plus 3, plus 4, plus 5, plus 6, plus 7, plus 8, plus 9. If you have done all of the steps correctly your answer should be 45. If you continue the process and to 45, add 1 plus 2, plus 3, plus 4, and so on up to 9, your next answer should be 90. Each time you complete a round from 1 to 9 your answer will increase by 45. When you have done it 20 times your answer will be 900. For variety, you might begin by setting the number 2 on your abacus. Then, to this add 1 plus 2, plus 3, plus 4, and so on up to 9. This will give you an answer of 47. One of the best ways for developing the skill of entering numbers automatically, whether by direct or indirect means, is to add the same number to itself again and again, such as 1 plus 1, plus 1, plus 1, plus 1, or, 7 plus 7, plus 7, plus 7, plus 7, on up to the addition of these numbers a hundred or more times.

THE ADDITION OF TWO-DIGIT NUMBERS

When adding a series of two-digit answers, users of Braille, pencil and paper, or other forms of written arithmetic do their addition from right to left. For example, in working the problem 18 plus 27 plus 36 equals 81, you would add first the 8 of 18, the 7 of 27, and the 6 of 36, for a total of 21. You would set down the 1 of 21 and carry the 2 to the next column. Adding the 2 of 21, the 1 of 18, the 2 of 27, and the 3 of 36, for a total of 8. This 8 would be set immediately to the left of the 1 obtained at the end of the first column of figures, and this would give the final total of 81. Using the abacus, addition is done from left to right, and there is no need to carry any numbers at all. Let's do this same problem on the abacus. Eighteen plus 27 plus 36 equals 81. On two columns, which we will call a b, we set the number 18; setting 1 on a and 8 on b. Next on a b we set the number 27. Set 2 on a and 7 on b. We can set the 2 on a directly, but it is not possible to set the 7 on b by direct means. So, to do this, we have to apply the second secret for adding the number 7. The secret is to add 7, clear 3, set 1 left. So, remove 3 from column b, and set 1 on column a. This now gives 45 set on a b. Finally, to 18 plus 27, we want to all 36. Set 3 on a and 6 on b. In order to add 3 to a we must use the first secret for adding 3. This is to add 3, set 5, clear 2. This gives us 7 on a and 75 on a b. In order to add 6 to b we must use the first secret for the addition of the number 6. To add 6, set 1, clear 5, set 1 left. So, then, on b make one upward motion with the index finger which sets 1 and clears 5. Then shift your attention to column a, and set 1. This gives you 8 on a and 81 on a b, which is our final total. Now, all of this may seem to be unduly complicated, but remember we are going step-by-step through a process which is actually easier to do than to talk about. When you have committed the secrets to memory, or, have so completely understood the theory of their operation as to make your finger movements automatic, you will be able to enter and change numbers without conscious thought or any mental effort at calculation.

Because of the nature of its operation, the abacus is not a recording device, that is, it is not possible to record the step-by-step process we go through in order to arrive at a final answer. All we have on the abacus is a final result. This means there is no way for us to check our work except to do it again. However, the speed with which the abacus allows us to work problems in arithmetic more than makes up for this apparent disadvantage.

From the example which we just gave, that is, 18 plus 27, plus 36, you should note that as we entered numbers on the abacus, we automatically calculated them. Eighteen plus 27 is 45, and 45 plus 36 is 81. So, that as you work along you always have a running total of your work, with the final answer forming automatically as you finish the last step of the process. In order to take advantage of the automatic nature of the abacus, you must have a thorough knowledge of the secrets of addition and must have complete skill in each of the methods for setting the digits from 1 to 9. Once you have this skill, the abacus' action is so automatic that it will be possible for you to have a friend set a number on the abacus without telling you what that number is. Then, you could be asked to add a second number to this first number, which is unknown to you, and if you enter the number correctly, you would automatically have the correct answer, even though you still have no idea of the original number set by your friend.

PRACTICE EXERCISES

In order to build skill, begin by repeatedly adding such numbers as 11 plus 11, plus 11, plus 11, etc., or, again, 88 plus 88, plus 88, plus 88, until your fingers move automatically and set digits correctly. You may recall that in adding single digit numbers we add 1 plus 2, plus 3, plus 4, and so on up to 9 and obtained the total of 45. Now, try adding 11 plus 22, plus 33, plus 44, plus 55, plus 66, plus 77, plus 88, plus 99. If you do all of the steps correctly, when you have finished setting 99, your abacus should show a total of 495. Vary the operation by clearing your abacus and trying it again, this time starting with 99 plus 88, plus 77, plus 66, and on down to plus 11 — at which point you should again have a total of 495. But, the method of setting numbers will be different. Remember *practice makes perfect*, and before going on with the study of the abacus, be sure that you understand and can do perfectly everything we have covered so far.

THE ADDITION OF NUMBERS HAVING THREE OR MORE DIGITS

Numbers having three or more digits, regardless of how many, are added in exactly the same way as two-digit numbers, that is, the addition is done from left to right, and the only special care that is needed is to be sure that each digit is set on a column where it belongs. Let us work at the right end of the abacus, and if we are adding two three-digit numbers, we will use the three right-most columns on the abacus. Set the digits, and it may well be that as we set our numbers our work will spill over onto the column immediately to the left, in other words, making use of the four right-most columns on the abacus. If we have a seven-digit number, and to it we want to add a three-digit number, we set the three-digit number in its proper place, on the three columns at the right end of the abacus, and the answer will form automatically.

ADDING SUMS OF MONEY

When adding sums of money, which contain both dollars and cents, you should use your unit mark. Imagine the unit mark as the decimal point separating the cents from the dollars. The amount $347.18 would be written in this way: 3, 4, 7, with the 7 on a column immediately to the left of the unit mark; 1, 8, with the 1 of 18 on the column immediately to the right of the unit mark. If, to this amount, we want to add $7.50, we use three columns, setting the 7 of $7.50 on the column immediately to the left of the unit mark, and if to this amount we want to add $9.00, we set the 9 on this same column, that is, the column immediately to the left of the unit mark, because we are actually adding 9.00.

PRACTICE EXERCISES

On your abacus set 789. To this, add 789, plus 789, plus 789, plus 789 until you have set this number 789 for a total of nine times. The correct answer to this problem of addition is 7,101. For another exercise, add 111 plus 222, plus 333, plus 444, plus 555, etc., until you have 999. Your answer should be 4,995. Again, on your abacus set 5,049 . This is a four-digit number. To this number add 111 plus 222, plus 333, etc., until you have added all of the numbers up to and including 999. These are three-digit numbers and should be entered correctly on the three right-most columns. If you do all the steps correctly, your final answer will be 10,044. Another interesting exercise is to treat as one cycle the numbers 123 plus 456, plus 789. Repeat this cycle nine times and you will obtain a final result of 12,312.

Before you go on in this work, be sure you understand what has been covered so far. Be sure, too, that you are able to do all the operations which we have outlined so that you will be in a better position to understand and to do the steps which will follow.

MULTIPLICATION

If you have practiced enough so that you are now able to do addition on the abacus comfortably, and are able to understand what you are doing, you are now ready to begin the study of multiplication.

Multiplication is really a rapid form of addition. For example, 9 multiplied by 6 is 54. This is the same as adding 9 plus 9, plus 9, plus 9, plus 9, plus 9. Thus, to multiply a number by 3 is to add that number three times; to multiply it by 74 is to add the same number seventy-four times, etc.

Each part of a problem in multiplication has a name, and in order to discuss multiplication intelligently, we should learn these names so that we will all be speaking the same language. The number which is multiplied is the *multiplicand*. The number by which the multiply is done is the *multiplier*, and the answer is called the *product*. In the example 6 multiplied by 3, 6 is the multiplicand, multiplied by 3, the multiplier, and it gives us a product of 18.

In working out a problem in multiplication on the abacus the procedure is quite simple. The multiplicand is multiplied by the multiplier and the product is set in the correct position to the right of the multiplicand. If the multiplicand is a number having more than one digit, each digit of the multiplicand is multiplied by all the digits of the multiplier, and the product of all these steps of multiplication is set on the abacus to the right of the multiplicand in accordance with rules of positioning which will be discussed later. When a digit of the multiplicand has been multiplied by all the digits of the multiplier, it is cleared and our attention is centered on the next digit to the left. When we have completed multiplying all the digits of the multiplicand by all digits of the multiplier, the multiplicand will have been cleared, and our total answer or grand product will have formed automatically upon the abacus board.

For our first example of multiplication, let's multiply a one-digit number by another one-digit number. Six multiplied by 7 is 42. On the first column at the extreme left-hand end of the abacus set the multiplier 7. Next, leave two columns unused and set the multiplicand 6. This means that 7 should be set on column a, columns b and c will be unused, and the multiplicand 6 will be set on d. In all the examples which we shall give, we will assume that the multiplicand is set up in such a way that its last digit will occur on a unit column, that is, on a column immediately to the left of a unit mark. So, whenever we say that the multiplicand is set on column d, or on columns d e f, we will assume that the last digit of the multiplicand is set upon a unit column. Bear this in mind and follow this practice. Now, in multiplying 6 by 7 we have already learned the multiplication table, so we automatically set the product 42 immediately to the right of the multiplicand, or on columns e f.

If you have a good memory it is not necessary to set the multiplier of every problem, but you should know how to set down an entire problem so that you will be able to do it when working with extremely complex problems. Now, let's do a problem having one-digit multiplicand and a two-digit multiplier. The problem: 7 multiplied by 42 is 294. First, set the multiplier 42 on columns a b. Leave c and d unused. Then, set the multiplicand 7 on column e which must be a unit column. Now, we are ready to begin multiplying. The first step is to multiply the 7 by the 4 of the 42. Seven by 4 is 28. So, set 28 on columns f g. Next, multiply the multiplicand 7 by the 2 of 42. Seven by 2 is 14. Set the 1 of the 14 on column g and the 4 on column h. Next, clear the multiplicand 7 from column e and the job will be done, leaving the product 294, on columns f g h.

DISCUSSION

Notice that the first time we multiplied the multiplicand 7, we multiplied it by the 4 of the 42, giving us the product of 28. When this was done, we then multiplied it by the 2 of 42, giving us the product of 14. By multiplying in this way, that is, by applying the digits of the multiplier in the order of their occurrence, we do away with the need to carry numbers, and our product is formed automatically. Notice, too, the way in which the digits of the product were entered onto the abacus. First, we multiplied the multiplicand 7 by 4 of 42. Seven by 4 is 28, and this product was set down immediately to the right of the multiplicand, or on columns f g. Then, we multiplied this 7 by the 2 of 42. Seven by 2 is 14. The 1 of the 14 was set on column g and the 4 on column h. In other words, the 8 of 28 and the 1 of 14 overlapped. This is an extremely important concept for you to understand. The first time we multiply the multiplicand by a digit of the multiplier, the resulting product is entered upon the first position. Then, the product resulting from the second multiplication of the multiplicand is entered upon the second position. Seven by 4 is 28, first multiplication, first position. Seven by 2 is 14, second multiplication, second position.

THE POSITION CONCEPT

A position consists of two columns. Exactly which column will depend upon the position's rank, that is, whether it is the first position, the second position, third position, and so on. The first position is made up of columns 1 and 2 immediately to the right of that digit of the multiplicand to which the position belongs; the second position is made up of columns 2 and 3; the third position, columns 3 and 4; the fourth position, columns 4 and 5; and so on. From this, you can see that, if it were necessary to multiply a digit in the multiplicand 27 times, the twenty-seventh position would be columns 27 and 28 to the right of that digit. The first time that a digit of the multiplicand is multiplied, the resulting product is entered in the first position. Product of the second multiplication is entered into the second position; third multiplication, third position; fourth multiplication, fourth position. You have not yet heard the entire story of positions, but this is as far as we should go for the present. Let's firm up our understanding with another example.

Five multiplied by 37 is 185. On columns a b, set the multiplier 37. Leave c and d unused, and set the multiplicand 5 on column e, which must be a unit column. Now, multiply the multiplicand 5 by the 3 of 37. Five by 3 is 15. So, set 15 in the first position, on columns f g. Next, multiply the 5 in the multiplicand by the 7 of 37. Five by 7 is 35. Set 35 on columns g h. Cancel the multiplicand 5. The job is done, and the correct product 185 will be found on columns f g h.

DISCUSSION

The first multiplication of 5 resulted in a product of 15. This was entered into the first position. The product of the second multiplication of 5, this time by the 7 of 37, was 35. The second multiplication, therefore, enter into the second position. Earlier, we said that a thorough working knowledge of the secrets of addition was essential before going on with the study of the abacus. Let's show you what we mean by another example. This time, multiply 6 by 37. Enter the multiplier 37, on columns a b. Leave c and d unused, and set the multiplicand 6 on column e, again a unit column. Six by 3 is 18. So, set the product 18 on columns f g. Next, 6 by 7 is 42. So set the product 42 on columns g h. Finally, clear column e of the multiplicand 6, and the job is done, leaving the correct product of 222 on columns f g h.

DISCUSSION

Six by 3 was the first multiplication, and its product 18 was entered into the first position. Six by 7 was the second multiplication of the multiplicand, and the product 42 was entered into the second position. But, in order to enter the 4 of 42 on column g, the column containing the 8 of 18, it was necessary to make use of one of the secrets for the addition of 4. Because, by entering 4 upon a column already containing the number 8, we were adding 4 to 8. The only secret which we could possibly have applied was to add 4, clear 6, set 1 left.

When we left our discussion of the position concept, we said that we had not told the entire story. You will note that in all the examples we have used so far, every time we multiplied the multiplicand by a digit of the multiplier, we obtained a two-digit sub-product. We have said that a position consists of two columns. So, if we have two columns and two digits, everything should come out even. But what happens when we have a situation in which multiplication results in a one-digit product? Let's do an example and find out. This time, 4 multiplied by 42 equals 168. On columns a b, set the multiplier 42. Leave c and d unused, and set the multiplicand 4 on the unit column e. Four by 4 is 16. So, set the product 16 on columns f g. Four by 2 is 8. So, set 8 on column h. Then clear column e of its multiplicand 4, and the job is done, leaving 168 on columns f g h.

DISCUSSION

We have said that when a digit of the multiplicand is multiplied by a digit of the multiplier, the resulting product must be entered in its appropriate position. When this product is a two-digit number we have no problem since we use both columns of the position. When the product is a single-digit number, we will have no problem either, if we will remember that *singlè-digit products* are entered upon the *second column* of the position.

In the example we just did, 4 by 42, 4 by 4 was 16—first multiplication, first position, resulting in 16 being entered on columns f g. Then, 4 by 2 was 8—the second multiplication, therefore, second position. But, 8 is a single-digit number. Therefore, it must be entered upon the *second column* of its position.

There is a simple technique for locating not only the correct position but the correct column within a position for the entry of a given digit. Set up your abacus with 42 on a b, leave c and d unused, and set 4 of column e. The digit of the multiplicand with which you are working at a given time is called the *marker column*, and positions are marked off to the right of this column. Locate column e, the marker column, with the index finger of your left hand. Now, place the index finger of your right hand on column f and the middle finger of your right hand on column g. The index and middle fingers of your right hand are now resting upon the first position. Move this pair of fingers one place to the right. The index finger should be on column g, the middle finger on column h. Observe that this is the second position, and that the middle finger of your right hand is resting upon the *second column* of this position. In actual practice, then, when the product to be entered on a given position is a two-digit number, the first digit will be entered upon the column on which your index finger rests, and the second digit will be entered upon the column touched by your middle finger. In the same way, if the product is a single digit number its entry will be made upon a column on which your middle finger rests because your middle finger will always rest upon the *second column* of a position.

So far we have worked only with one-digit multiplicands. Now, let's try an example with a three-digit multiplicand but a one-digit multiplier. Eight-hundred-nineteen by 9 equals 7,371. On column a, set the multiplier 9. Leave b and c unused, and set the multiplicand 819 on columns d e f with f as a unit column. Next, multiply the 9 of 819 by the multiplier 9. Nine by 9 is 81. Set this product on g h. Next, cancel the 9 of 819. Now multiply the 1 of 819, on column e, by 9. One by 9 is 9, and this product should be set on column g. Next, clear e of the 1 of 819, and multiply the 8 of 819 by 9. Eight by 9 is 72. This is to be set on columns e f. Clear d of its 8, and the job is done, leaving the correct product 7,371 on columns e f g h.

DISCUSSION

When we multiplied the 9 of 819 by 9 we had a product of 81. This was the first, and only, multiplication of 9. So, it was entered into the first position, or upon columns g h. Then, we cancelled the 9 in the multiplicand. Next, we multiplied the 1 of 819 by our multiplier 9. One by 9 is 9. This was the first multiplication of this digit of the multiplicand, and it resulted in a one-digit product, which is to be entered into the first position. Remember, though, that one-digit products are entered upon the second column of their positions. The second column of the first position for the multiplicand digit 1 is column g. So, we enter the 9 on column g. To do this, you must use a secret of addition, which says, to add 9, clear 1, set 1 left. So, remove 1 from column g and set 1 on column f, leaving 171 on columns f g h. Next, clear e of its 1. Now, we are ready for the final step, 8 of 819 multiplied by our multiplier 9. Eight by 9 is 72, and this is set on columns e f in a straight forward manner. Clear d of its 8, leaving 7,371 on e f g h.

Now let's do an example having a two-digit multiplier and a three-digit multiplicand. The problem 819 multiplied by 23 equals 18,837. On columns a b, set the multiplier 23. Leave c and d unused, and set the multiplicand 819 on columns e f g, with g as a unit column. Multiply the 9 of 819 by the 2 of 23. Nine by 2 is 18. Set this product on columns h i. Next, multiply the 9 of 819 by the 3 of 23. Nine by 3 is 27, and this product is to be set on columns i j. Next, clear g of its 9, and shift your attention to column f. Multiply the 1 of 819 by the 2 of 23. One by 2 is 2, and this product should be set on column h. Next, multiply the 1 by 3. One by 3 is 3 and this product belongs to column i. Then, clear 1 from column f and turn your attention to column e which contains the 8 of 819. Eight by 2 is 16. So, set 16, on columns f g. Eight by 3 is 24. Set this product on g h. Clear e of its 8, and the job is done, leaving the product 18,837 on columns f g h i j.

DISCUSSION

Our first step was to multiply the 9 of 819 by the 2 of 23. Nine by 2 resulted in a product of 18. This was the first multiplication of 9, and 18 was entered in the first position. We next multiplied 9 by the 3 of 23, giving us the product of 27. This was the second multiplication of 9, so, entry of this product was in the second position. To enter the 2 of 27 upon the column containing the 8 of 18, it was necessary to apply that secret for the addition of 2 which says, to add 2, clear 8, set 1 left. At this point our 9 has been multiplied by all the digits of the multiplier. So, 9 is ready to be cleared, and we are ready to turn our attention to the digit 1 of 819. One by 2 is 2. This is the first multiplication of the digit 1, and the resulting product is to be entered upon the first position. Next, 1 by 3 is 3. This is the second multiplication of 1, therefore, entry is upon the second position, and again, since 3 is a one-digit number, it is entered upon the second column of its position. One, having been multiplied by all digits of the multiplier, is ready to be cleared. Next, we multiply the 8 of 819 by 2. Eight by 2 is 16—first multiplication, first position. Eight by 3 is 24—second multiplication, second position. The 2 of 24 can be entered by direct means, but to enter 4, we must use that secret for the addition of 4 which says, to add 4, set 5, clear 1. Now that the 8 has been multiplied by all digits of the multiplier, it may be cleared, leaving our product 18,837 on f g h i j. Remember that, regardless of the number of digits in the multiplier or the multiplicand, if entry of products is made upon the proper positions and if the correct secrets for the addition of numbers are applied, your multiplication will be done automatically, and with absolutely no mental effort on your part. Of course, you must have a thorough knowledge of the multiplication tables up to and including 9 by 9. Now let's try one more example. We'll use the same multiplicand, 819, but this time our multiplier will be a three-digit number, 123, and our answer, or product, will be 100,737. On columns a b c, set the multiplier 123. Leave d and e unused, and set 819, the multiplicand, on f g h, with h as a unit column. Now, multiply the 9 of 819 by 1. Nine by 1 is 9. So, set 9 on column j. Next, multiply the 9 by the 2 of 123. Nine by 2 is 18. Set 18 on j k. Finally, multiply the 9 by the 3 of 123. Nine by 3 is 27, and this product is entered upon k l. Now, clear 9 from column h and turn your attention to column g. Multiply the 1 of 819 by the 1 of 123. One by 1 is 1. So, set this product on column i. Next, multiply 1 by 2. One by 2 is 2. This is to be set on column j, and finally, 1 by 3 is 3, and this product is to be set on column k. Now, clear column g of its 1 and turn your attention to column f. Multiply the 8 of 819 by the 1 of 123. Eight by 1 is 8. Set 8 on column h. Next, 8 by 2 is 16. This is to be set on h i, and finally 8 by 3 is 24, which is to be set on i j. Clear f of its 8 and the job is done, leaving 100,737 as the product on g h i j k l.

DISCUSSION

The first time we multiplied the 9 of 819 it was multiplied by 1, giving a product of 9. This was to be entered upon the first position. Nine is a one-digit number; therefore, entry should be made upon the second column of the first position. Next, 9 by 2 gives us a product of 18—second multiplication, therefore, second position. To enter 1 of 18 on the column already containing the 9 of our previous multiplication, it is necessary to use the secret for the addition of 1, which says, to add 1, clear 9, set 1. The 8 of 18 can then be entered in a straightforward way on the second column of its position. Next, 9 by 3 is 27. To enter 2, we must use the secret for addition which says, to add 2, clear 8, set 1 left. Seven can be entered directly. This was the third multiplication, therefore, entry was upon the third position, and we cleared our 9 from the multiplicand. Next, 1 by 1 gave a product of 1—first multiplication, first position. But, since 1 is a one-digit number, it was entered upon the second column of the first position. Next, 1 by 2 is 2—second multiplication, therefore, entry is made upon the second position again, on the second column of the position, because 2 is a one-digit number. Next, 1 by 3 is 3—third multiplication, third position, but the second column of this position because 3 is a one-digit product. Now, we clear 1 from the multiplicand and turn our attention to the 8 of 819. Eight by 1 is 8—first multiplcation, so entry is made in the first position. But, since 8 is a one-digit product, it is entered upon the second column of this position. Next, 8 by 2 is 16—second multiplication, second position. Finally, 8 by 3 is 24. To enter the 2 of 24 on the column which now contains an 8, we must use that secret for the addition of 2 which says, to add 2, clear 8, set 1 left. So, we clear the 8, but looking at the column immediately to its left we see that it contains a 9. So, we clear this 9, and set 1 to its left, or on column g. Finally, to enter the 4 of 24, we must use the secret for the addition of 4 which says, to add 4, set 5, clear 1. This is done on column j. We then clear 8 from the multiplicand, and our product, 100,737 is left on columns g h i j k l.

In all the problems which we have done, we have set both the multiplier and the multiplicand. We have said that some people might prefer to set only the multiplicand and retain the multiplier in memory. If you would like to try this, feel perfectly free to experiment. If you do experiment along these lines, we must insist that you adhere to the placement of the last digit of the multiplicand upon a unit column, that is, upon a column immediately to the left of one of the unit marks. Doing this will save you both time and effort when trying to locate the last digit of your product in case the product contains a number of zeros.

DETERMINING THE LENGTH OF THE PRODUCT

To show what we mean by this, consider the problem 125 multiplied by 8 equals 1,000. The number 1,000 is made up of the digits 1, 0, 0, 0. But, on your abacus a zero may easily be confused for an unused column. However, this confusion is not possible if you place the last digit of the multiplicand upon a unit column and observe the following rule. The last digit of the product falls to the right of the last digit of the multiplicand by as many places as there are whole-number digits in the multiplier plus one additional place because this is a problem in multiplication. To see this rule in operation, set your multiplicand 125 in such a way that the 5 of 125 falls on a unit column, that is, a column immediately to the left of a unit mark. Let's call this column, column e. Now, multiply 125 by 8, following the procedures you have already learned. Five by 8 is 40—first multiplication, first position. Two by 8 is 16—first multiplication, first position. To enter 6, use the secret to add 6, clear 4, set one left. One by 8 is 8—first multiplication, first position. To enter 8, use the secret which says, to add 8, clear 2, set 1 left. Of course, it goes without saying that after each multiplication you have cancelled that digit of the multiplicand which you have just multiplied. Now, the abacus should be perfectly clear except for the digit 1, occurring on a certain column; it doesn't matter which one at the present time. We know that 125 multiplied by 8 is not 1, but just how many 0's follow this 1? Run your finger along the face of the lower edge of the frame until you find the unit mark to the left of which you have just set the 5 of 125. The dot immediately to the left of this unit mark will then mark the column on which this 5 has been set. Place your finger upon this column and then move it to the right first one place because there is just one whole number digit in the multiplier, and then one additional place because this is a problem in multiplication. Your finger should now be resting on column g. This is not an unused column then, but a column which contains the digit 0. So, g is one, moving to left, f is two, e is three 0's, and column d contains a 1. So, the answer is 1 followed by three 0's, or 1,000. This should give you the idea of the importance of placement of the last digit of the multiplicand upon a unit column. But, the subject will receive more thorough discussion when we talk about the multiplication of decimals.

THE TREATMENT OF ZEROS

Suppose you have a multiplier such as 3,006, and the multiplicand is 7. The first multiplication gives a product of 21; 7 by 3 is 21. Seven by 0 is 0, 7 by 0 is 0, 7 by 6 is 42. This product, 42, must be entered into the fourth position. In other words, when zeros occur in the middle of a multiplier, they must be treated as any other digit so far as the position concept is concerned.

Before going on in this work, be sure you understand thoroughly and can perform all of the operations learned so far in the study of addition and multiplication.

PRACTICE EXERCISES

For practice, multiply the number 123,456,789, a nine-digit number, first by 18, then by 27, 36, 45, 54, 63, 72, 81, and 99. Then, practice again. This time multiply 987,654,321 by 18, 27, 36, 45, 54, 63, 72, 81 and 99. We can also make use of some of our problems in addition. Seven hundred eighty-nine multiplied by 9 will give us 7,101.

SUBTRACTION

Subtraction is the process which we use in order to compare two numbers for the purpose of learning the difference between them. If we want to know the difference between 375 and 975, we subtract 375 from 975, and we learn that the difference is 600. The various parts of a problem in subtraction are named and, in order to avoid confusion, we should learn these names before going any further. The number from which a number is subtracted is called the *minuend*. The number which is subtracted is called the *subtrahend*, and the difference or answer which results is called the *remainder*. In our 375 and 975 example, 975, the number from which the subtraction is done, is the minuend; 375, the number which is subtracted, is the subtrahend; and the difference, 600, is the remainder.

In written arithmetic, subtraction is done from right to left. For example, in written arithmetic the 5 of 375 would be subracted from the 5 of 975. Then, the 7 of 375 would be subtracted from the 7 of 975. Then, and finally, the 3 of 375 is subtracted from the 9 of 975, leaving the remainder of 600. When we use the abacus, however, we work from left to right, just as we do in all the processes of arithmetic. So, we must set 975 on the abacus, and from the 9 we would subtract 3, from the 7 we would subtract 7, and from the 5 we would subtract 5, and automatically form 600 as our remainder.

As in the case of addition, when a number is to be subtracted directly, the newcomer to the use of the abacus is presented with no serious problem. For example, it's no trick at all to subtract 2 from 7. All you need do is move the two one-unit counters away from the bar. This leaves a remainder of five. But, suppose you wanted to subtract 3 from 7. This becomes another story. It is not possible to move three one-unit counters away from the bar, and so we must resort to an indirect method of subtraction. As in the case of addition, we must make use of the system of complementary numbers or, if you prefer, you may memorize the secrets of subtraction instead. In order to subtract 3 from 7, it would be necessary to set 2 and clear 5. Suppose I have 7 cents, 1 five-cent piece and 2 one-cent pieces in my hand, and I want to subtract 3 cents from this amount. If you gave me 2 cents, and I, in turn gave you my five-cent piece, we would have subtracted 3 cents from my 7 cents, leaving me with the remainder of 4 cents. If we learn the secrets of subtraction, or, if we learn to apply a principle of the use of complementary numbers so quickly that it becomes automatic, we can subtract with absolutely no mental effort. As in the case of the other processes, subtraction would be a purely mechanical operation. There are seventeen secrets for subtraction, just as there are seventeen secrets for addition. They are as follows:

To Subtract	Secret
1	Clear 1 left, set 9
1	Set 4, clear 5
2	Clear 1 left, set 8
2	Set 3, clear 5
3	Clear 1 left, set 7
3	Set 2, clear 5
4	Clear 1 left, set 6
4	Set 1, clear 5
5	Clear 1 left, set 5
6	Clear 1 left, set 4
6	Clear 1 left, set 5, clear 1
7	Clear 1 left, set 3
7	Clear 1 left, set 5, clear 2
8	Clear 1 left, set 2
8	Clear 1 left, set 5, clear 3
9	Clear 1 left, set 1
9	Clear 1 left, set 5, clear 4

In using these secrets, when we speak of clearing 1 left, we mean remove 1 one-unit counter from the first column immediately to the left of the column on which our subtraction is being done. If this column happens to contain a 0, it will, of course, be impossible to remove 1 from this column. So, you should move over to the left one more column. Remove 1 from this column, and change the 0 to a 9. In the case of two or three or more 0's, continue moving to the left until you find a column from which it is possible to remove 1. Remove the 1 and change all of the intervening 0's to 9's. If you learn to do addition and multiplication efficiently, subtraction should come to you quite easily. It will just be a matter of learning to reverse the secrets of addition. By now, you should have enough skill in using the abacus to make the use of numerous examples unnecessary. However, we will show one problem in subtraction step-by-step, and let you, the reader, take it from there. The problem: 1,234 minus 567. Use the four right-most columns on the abacus, and we will call them a b c d. On a b c d, set 1,234. From the 2 of 1,234, subtract the 5 of 567. Five cannot be subtracted directly from the 2, so we must use the secret for subtracting 5, which is, to subtract 5, clear 1 left, set 5. So, from column a, clear 1 and set 5 on b. On b c d, we now have 734. From the 3 of 1,234, subtract the 6 of 567. Again, 6 cannot be subtracted from 3 directly. So, we must use the second secret for subtracting 6, which is, to subtract 6, clear 1 left, set 5, clear 1. So, clear 1 from b and set 5, clear 1 on column c. This changes the 7 on b to 6 and gives us 674 on b c d. Now, from the 4 of 1,234, subtract 7 of 567. In order to do this, you must use that secret for the subtraction of 7 which says, to subtract 7, clear 1 left, set 5, clear 2. Clear 1 from c, and on d, set 5, clear 2. This gives you the final answer, or remainder, of 667 on b c d. To verify this, to the 667 on b c d, add 567. You will see that your answer is 1,234.

PRACTICE EXERCISES

For practice in subtraction, we suggest that you reverse those exercises given as practice for addition. For example, on the abacus set 7,101, and from this subtract 789, minus 789, minus 789, etc. When you have repeated this subtraction nine times, your answer should be 0. Or set 45, and from it subtract 1, minus 2, minus 3, minus 4, minus 5, minus 6, minus 7, minus 8, minus 9. Your answer should again be 0. If you are subtracting a three-digit number from a four- or five-digit number, it is important that you do your subtracting from the appropriate column. In subtracting 789 from 7,101, the 7 of 789 should be subtracted from the column which holds the first 1 of 7,101. In subtracting 1 from 45, 1 is subtracted from the column containing the 5 of 45.

We are now ready to begin the study of the process of division, but, before you go on to learn anything new, please review and be sure you understand thoroughly all of the processes which we have learned so far.

DIVISION

Just as multiplication is a rapid form of addition, so division is the rapid form of subtraction, that is, by division, we are able to tell how many times one number can be subtracted from another. For example, 74 divided into 1,480 gives an answer of 20. This means that the number 74 can be subtracted from the number 1,480 a total of twenty times. Similarly, 74 into 1,485 goes twenty times with a remainder of 5. This is the same as saying 74 can be subtracted from 1,485 twenty times leaving a remainder of 5. This means, of course, that where it can be used, division is much faster, much more accurate than repeated subtraction, just as multiplication is much quicker and easier than repeated addition.

Just as in the case of multiplication and subtraction, various parts of a problem in division have names, which we should learn before actually studying the process of division itself. The number which does the dividing is the *divisor*. The number into which it is divided is the *divdend*, and the answer is the *quotient*. Thus, in the problem 1,480 divided by 74 equals 20, 74 is the divisor, 1,480 is the dividend, and 20 is the quotient.

When setting up a problem in division on your abacus, the divisor is set at the extreme left end of the board. To the right of the divisor four columns are left unused, and then the dividend is set. Just as in the case of multiplication, once you have gained skill in the use of the abacus, you may prefer not to set down the divisor at all, but to keep it in your memory. In such a case, leave the two left-most columns on your abacus unused, and begin setting the dividend on the third column, that is, leave columns a and b unused, and set the first digit of the dividend on column c. At *least* two unused columns must be provided at the left end of the abacus, but in any case enter the dividend in such a way that its last digit falls upon a unit column, that is, a column immediately to the left of one of the little unit marks found on the face of the bottom member of the abacus frame.

In division, the quotient figures will appear to the left of the dividend, and this is why at least two unused columns must be provided. In the examples that we shall do, we shall set both the divisor and the dividend just to show you how it is done. Assume always that the last number of our dividend is set upon a unit column.

SHORT DIVISION

Short division is a process by which a dividend, made up of any number of digits, is divided by a divisor having only one digit. The process is worked by comparing the divisor with one or more digits of the dividend. This results in a quotient figure which is entered on the abacus. This quotient figure is then multiplied by the divisor, and the product resulting from this multiplication is subtracted from the dividend. The subtraction is done following rules of positioning similar to those used in multiplication. These will be discussed later.

Before going any further, let us do the example 736 divided by 4. To do this, we will use columns a through h. Set the divisor 4 on column a, leave b c d e unused, and set the dividend, 736 on columns f g h. We assume h to be a unit column. Now compare the 4 which is our divisor with the 7 of 736. Four will go into 7 just 1 time. So, our first quotient figure is 1. Set this quotient figure 1 on the second column to the left of the 7 of 736, in other words on column d. Next, multiply the quotient figure 1 by the divisor 4. One by 4 is 4. So, subtract this 4 from column f, or from the 7 of 736. This leaves 3 on column f, and 336 on f g h. Next, compare the divisor 4 with the 3 of column f. Four will not go into 3. So, you must compare this 4 with the first two digits of the dividend, that is, with the 33 on columns f g. Four will go into 33 eight times. So, our next quotient figure is 8. This is to be set immediately to the left of column f, or, on column e. On columns d e, you now have 18, and on columns f g h 336. Next, multiply this quotient figure 8 by the divisor 4. Eight by 4 is 32. So, subtract 32 from columns f g. This leaves 1 on column g, and 16 on g h. Since 4 is larger than 1, we must compare our divisor 4 with the 16 remaining on g h. Four into 16 goes four times. So, set the next quotient figure 4 on column f, immediately to the left of the digit 1 on column g. Multiply this quotient figure 4 by the divisor 4. Four by 4 is 16. Subtract this product 16 from the 16 on g h. This clears away the dividend, and leaves the final quotient or 184, on columns d e f. To varify your results, multiply 184 by 4 and you will obtain a product of 736.

DISCUSSION

You will remember that when we compared our divisor 4 with the 7 of 736, we set our quotient figure 1 two columns to the left of the dividend. But, later when we compared 4 with 33, and again with 16, we set the resulting quotient figures immediately to the left of the dividend. This was no accident. The rule for the placement of the quotient figure in short division is as follows: When the divisor is a number which is smaller than, or equal to, the first digit of the dividend with which it is being compared, the resulting quotient figure must be placed on the second column to the left of the dividend. In other words, the quotient figure and dividend must be separated by one unused column. When the divisor is larger than the first digit of the dividend to which it is being compared, the resulting quotient figure is set immediately to the left of the dividend. In this case, there is no unused column between quotient figure and dividend.

Review the example and you will see that, in the first comparison, 4 was smaller than the 7 of 736. So, the quotient figure 1 was separated from the dividend 7 by one unused column. In the second comparison, the 4 was larger than the 3 of 33, so the quotient figure 8 was set immediately to the left of this first 3. Finally, 4 is larger than the 1 of 16. So, again, the quotient figure 4 must be set immediately to the left of the 1 of 16.

We have worked one example and discussed it in great detail. We will now do another without benefit of follow-up discussion. Our problem: 738 divided by 6. Set the divisor 6 on column a. Leave b c d e unused, and on f g h set the dividend 738. Again, we assume h to be a unit column. Compare the divisor 6 with the 7 of 738. Six will go into 7 once. So, set your quotient figure 1 on column d. Next multiply the quotient figure by the divisor. One by 6 is 6. Subtract the product 6 from column f. Next, compare the 6 of the divisor with the 1 on column f. Six will not go into 1. So, you must compare the 6 with the 13 on f g. Six will go into 13 twice. Set your quotient figure 2 on column e. Multiply the quotient figure by the divisor. Two by 6 is 12. Subtract the product 12 from columns f g. Next, compare the divisor 6 with the 18 on columns g h. Six will go into 18 three times. Set your quotient figure 3 on column f. Multiply the quotient figure by the divisor. Three by 6 is 18. Subtract the product 18 from the 18 on columns g h. This clears the abacus, and leaves the quotient 123 on columns d e f.

Before learning anything new, review that which you have learned so far, and make sure that you are completely comfortable with short division, subtraction, multiplication, and addition. Practice problems in short division and verify your results by muliplication. This will give you skill in short division, and sharpen skills already gained in multiplication. Divide your telephone number by the first digit of your house number. Or, divide your telephone number by 2, 3, 4, 5, 6, 7, 8, 9. Multiply your quotient by the divisor each time to check your accuracy.

Multiply your age in years at your last birthday by 365. And then, add a day for each leap year since your birth. This will give you your age in days at your last birthday. Divide this figure by 7, and this will give you your age in weeks. This will also give you lots of practice. An interesting short division problem is this one: Divide the ten-digit number 1,111,111,101 by 9. Then look at your quotient. Interesting, isn't it? Only you can be the judge of when you are ready to go on with the next step. Having decided that you can add, multiply, subtract, and do short division comfortably, you are now ready to take up the study of long division.

LONG DIVISION

This is the process by which a dividend made up of any number of digits is divided by a divisor made up of two or more digits. In setting a long division problem on your abacus, you proceed just as in the case of short division. Make sure that the last digit of the dividend is set upon a unit column. The actual process for division is essentially the same, although it requires a few additional steps. Usually, the first digit of the divisor is compared with the first digit of the dividend. The resulting quotient figure is entered upon the appropriate column, and this quotient figure is then multiplied by each digit of the divisor, and each product resulting from this multiplication is subtracted from appropriate columns of the dividend in accordance with the rule of positioning.

For our first example, let's divide 576 by 24. Set the divisor 24 on columns a b. Leave c d e f unused, and set 576 on g h i. We assume i to be a unit column. Next, compare the 2 of 24 with the 5 of 576. Two goes into 5 two times, giving you a quotient figure of 2. Set this quotient figure 2 on column e. Now multiply the quotient figure by the first digit of the divisor. Two by 2 is 4. So, subtract the 4 from column g. This leaves 1 on g and 176 on g h i. Next, multiply the quotient figure by the second digit of the divisor. Two by 4 is 8. So, subtract the 8 from column h, leaving 9 on h, and 96 on columns h i. Next, compare the 2 of 24 with the 9 of 96. Two into 9 goes 4 times. Set your quotient figure 4 on f. Next, multiply the quotient figure by the first digit of the divisor. Four by 2 is 8. Subtract this 8 from h and multiply the quotient figure by the second digit of the divisor. Four by 4 is 16. This is subtracted from the 16 on h i, clearing the abacus of its dividend, and leaving the correct quotient 24 on columns e f.

DISCUSSION

If we bear in mind the rule for the placement of the quotient figure which we learned in our study of short division, it is easy to see how this problem was worked through to a successful solution. In our first comparison with the 2 of 24 and the 5 of 576, 2 is smaller than 5, so that the resulting quotient figure was separated from the dividend by one unused column. Again, in the second comparison, the 2 of 24 was smaller than the 9 of 96, resulting in a separation of quotient figure and dividend by one unused column. The rule for quotient figure placement in short division says that when the divisor is smaller than, or equal to, the first digit of the dividend, the resulting quotient figure is separated from the dividend by one unused column. When the divisor is greater than the first digit of the dividend, there is no unused column separating the quotient figure from the dividend. This rule is all right as far as it goes, but for long division, it does not go quite far enough. In most cases, it can be taken at face value, but in some instances it needs to be carried further. This is because the divisor in a long division problem consists of two or more digits. This means that sometimes the first digit of the dividend and the first digit of the divisor may be the same. When this happens, the second digits of both the divisor and dividend must be compared. If these digits are equal, then, the third digits must be compared, etc., until a point of inequality is reached. It is at this time that the rule for the placement of quotient figure is applied. This rule has been stated previously, that is, if the digit in the divisor is smaller than the digit in the dividend to which it is compared, the quotient figure is separated from the dividend by one unused column. But if the digit in the divisor is larger than that in the dividend to which it is being compared, then the quotient figure is placed immediately to the left of the dividend with no separation by an unused column.

Consider the problem 225 divided by 25. Set 25 on a b. Leave c d e f unused, and on g h i, set 225. Again, i is a unit column. Now, compare the 2 of 25 with the 2 of 225. They are equal, and, on the face of it, you might assume that you set the quotient figure on column e, that is, separated from the dividend by one unused column. Before making a final decision, let's compare the second digits of the divisor and dividend. The 5 of 25 is larger than the second 2 of 225. So, we must obey the rule which says that, when the digit in the divisor is greater than the digit in the dividend to which it is being compared, the resulting quotient figure is set immediately to the left of the first digit of the dividend, and not separated by an unused column. So, in this case, we would set our quotient figure on column f. Our quotient figure in this case is 9. So, with this 9 set on f, let's multiply it by the first digit of the divisor. Nine by 2 is 18. So, subtract 18 from g h. This results in 45 being left on columns h i. Now we multiply the quotient figure 9 by the second digit of the divisor 5. Nine by 5 is 45. We subtract 45 from the 45 on h i, clearing the abacus, and leaving 9 as our quotient.

DISCUSSION

The thoughtful reader will wonder why we did not explain how we arrived at our quotient figure 9 while working the previous problem. This omission was deliberate in order to avoid confusion and the explanation is given here instead. Whenever you encounter a problem of this type, that is, one in which the first digits of the divisor and the dividend are the same but the second digit of the divisor is larger than the second digit of the dividend, a quotient figure of 9 stands a better than even chance of being correct. This is not an explanation, of course, but is a statement of a generalized rule of thumb. Now, let's set up the problem again and look at it scientifically. On a b, set 25, on g h i, 225. If you compare the 2 of 25 with the 2 of 225 you would correctly assume the resulting quotient figure to be 1. But, if you were to try 1, it would not be long befor your mistake would become apparent. So, compare the 2 of 25 with the 22 of 225. Two goes into 22 eleven times. But, 11 is a two-digit number. So, it must be rejected as a quotient figure since we develop our quotient figures one digit at a time, and verify them through multiplication by the divisor and appropriate subtraction from the dividend.

Now, for a minute, let's think of our divisor 25 not as the whole number 25, but rather as the decimal fraction 2.5, or, 2½, or, as a number lying somewhere between 2 and 3. We have said that comparing the 2 of 25 with 22 of 225 gives a quotient figure of 11. But, since we are now thinking of our divisor as a number which lies somewhere between the whole numbers 2 and 3, let's compare 3 with the 22 of 225. Three goes into 22 seven times. If we chose 7 as our quotient figure, no harm would be done, since, during the process of solution, this quotient figure would become corrected to 9 automatically. Just how this would come about will be described later. So, by comparing 2 with 22, we have 11, a quotient figure which is too large; and by comparing 3 with 22 we have 7, a quotient figure which is too small. In the case of 225 divided by 25, think of our divisor as a number half-way between 2 and 3, and the quotient figure should be a number half-way between 11 and 7, or in this case 9, which turns out to be correct.

It is not intended that you should go through mental gymnastics every time this kind of situation occurs. This explanation is given only to satisfy those who are interested in theoretical considerations. For the sake of automatic division, free from mental effort, we suggest that you apply the generalized rule of thumb, stated at the beginning of this explanation. The process of division can be just as automatic, just as free from mental effort as are addition, multiplication, and subtraction. If automatic division is your goal, as it should be, you must learn to understand and master two secrets of success. First, is the placement of the quotient figure upon the correct column. We have already covered this point. Once the quotient figure has been set on the abacus, it is multiplied by every digit of the divisor, and the products of this multiplication are subtracted in an orderly way from the correct columns of the dividend. This subtraction is done in accordance with the rules of positioning which we have already learned in multiplication.

When we set our quotient figure on the abacus and then proceed to multiply it by the various digits of the divisor, we are reducing our dividend through a combination of the processes of multiplication and subtraction. In such a case, the quotient figure is the multiplicand, and the divisor is the multiplier. In multiplication, remember that each time you multiply a digit of the multiplicand by the digits of the multiplier your product is set in the appropriate position. Suppose you have a one-digit multiplicand and a five-digit multiplier. The product which results from the first multiplication of this multiplicand is set in the first position; the product of the second multiplication, in the second position, and so on, until you multiply the multiplicand by the fifth digit of the multiplier. This is the fifth time the multiplicand is multiplied, and the product which results is set in the fifth position. The first position, of course, is made up of the first two columns immediately to the right of the multiplicand. The second position is the second and third; the third position, the third and fourth; and the fourth position, fourth and fifth column immediately to the right of the multiplicand. Thus, the digit of the multiplicand which is currently being multiplied serves as the *marker digit*, and positions may be determined by counting to the right from this marker column, just as in the case of multiplication: first position, columns one and two; second position, columns two and three; third position, columns three and four, etc. Now, in the case of division, the quotient figure represents the multiplicand, the divisor, the multiplier. So, having obeyed the rules for the proper placement of the quotient figure, the quotient figure is then multiplied by each digit of the divisor, and the resulting products are subtracted from the dividend according to these principles of positioning. First multiplication is subtracted from the first position; second multiplication is subtracted from the second position, etc. If we have a three-digit divisor, the third product will be subtracted from the third position, and the third position will be the third and fourth columns to the right of the quotient figure, which in this case serves as our marker column. Let's work through an example and follow it up with a discussion. For our problem we will divide 567 by 21. Set the divisor 21 on a b. Leave c d e f unused, and on g h i set 567, our dividend; i, again, is a unit column. Compare the 2 of 21 with the 5 of 567. This results in a quotient figure of 2, since 2 will go into 5 two times. Set 2, the quotient figure, on e. Next, multiply it by the divisor 21. Two by 2 is 4. So, subtract 4 from column g. Next, multiply the quotient figure 2 by the 1 of 21. Two by 1 is 2. So, subtract this 2 from h. Now, compare the 2 of 21 with the 14 on g h. Two will go into 14 seven times. So, set your quotient figure 7 on column f. Next, multiply this 7 by the 2 of 21. Seven by 2 is 14. Subtract it from the 14 on g h. Next, multiply 7 by the 1 of 21. Seven by 1 is 7. Subtract this from the 7 on i. This clears the abacus of the dividend, and leaves the quotient 27 on e f.

DISCUSSION

When we compared 2 of 21 with the 5 of 567 the 2 was smaller than the 5, so that the resulting quotient figure 2 was separated from the 5 by one unused column. This means that column e, the column on which the quotient figure 2 was set, became the marker column. We then multiplied this 2 on column e by the 2 in the divisor. Two by 2 is 4, and it represents the first multiplication of this quotient figure. This first multiplication must have its product subtracted from the first position. Four is a one-digit number. So it must be subtracted from the *second column* of the position. With column e as our marker column, we count to the right two columns. F is the first, g is the second. So we subtract our 4 from g. Next, we multiply our quotient figure 2 on column e by the 1 of 21. This results in a product of 2. This is the second multiplication of our quotient figure, and it must be subtracted from the second position. The second position is made up of the second and third columns to the right of our marker column, and since 2, the product to be subtracted, is a one-digit number, it must be subtracted from the second column of this position. So position two consists of columns two and three to the right of the marker column. Therefore, the single digit 2 must be subtracted from the third column. Counting to the right, we find that f is the first, g is the second, and h is the third. So, we subtract 2 from column h.

In our next comparison, the 2 of 21 is compared with 14 on g h. Since the 2 in our divisor is greater than the 1 in the dividend, we set the resulting quotient figure 7 on f, that is, immediately to the left of the dividend, with no separation by an unused column. The first multiplication of this quotient figure 7 is by the digit 2 of the divisor. Seven by 2 is 14, and this subtraction is from the first position. Column f is now our marker column, since this is the column on which our multiplicand is set. So subtract 14 from the first position, that is, from the two columns immediately to the right of this marker column. The two columns to the right of f are g and h. Subtract 14 from g h. Next, we multiply the 7 by 1 and obtain the product of 7. This is the second multiplication of our quotient figure 7, and it must be subtracted from the second position. But 7 is a one-digit number. So it must be subtracted from the second column of this position. The second position is columns two and three to the right of the marker column, or, in other words, columns h and i. Since the product is a one-digit number, we must subtract it from column i. Seven subtracted from 7 clears our abacus and leaves the quotient 27 on e f. Multiplication of 27 by 21 will give us 567 and will verify the correctness of our work.

An easy way to find your position is to place the index finger of your left hand on the marker column, the column on which the quotient figure under consideration is set. Then, place the index finger and middle finger of the right hand on the two columns to the right of this marker column. This is position one. Moving your fingers over one place to the right will place you on position two, and so on. The middle finger of your right hand will automatically rest on the second column of each position, and it is from this column that the single-digit product must be subtracted. Two-digit products must be subtracted from the first and second columns of the position, or, from the columns on which both the index finger and middle finger of your right hand are resting.

CORRECTING THE QUOTIENT FIGURE

In long division, we arrive at our quotient figure by comparing the first digit of the divisor with the first one or two digits of the dividend. Since the divisor consists of more than one number, the quotient figure obtained from only first-digit comparisons is subject to error. When such an error is made, the quotient figure must be corrected. If too small a quotient figure is chosen, it must be raised to the correct figure by *upward correction.* If too large a quotient figure is chosen, it must be reduced by means of a *downward correction.*

THE UPWARD CORRECTION

When too small a quotient figure has been chosen, the process by which it is corrected upward is *self-corrective* in nature. To find out just how it works, let's do again the problem 225 divided by 25. On a b set 25. Leave c d e f unused, and set 225 on g h i. You will remember that in the explanation following this problem, we said that if we thought of the 25 at 2.5, it would be a number which would occur somewhere between the whole numbers 2 and 3, and that if we compared 2 with 22 we would obtain a quotient figure of 11, which would not be useable because it is a two-digit number. On the other hand, if we chose to compare 3 with 22, we would obtain a quotient figure of 7, which would be too small. But, just for the sake of the example, let us set 7 as our first quotient figure. Set 7 on column f. Multiply 7 by the 2 of 25. Seven by 2 is 14. So subtract 14 from g h. Next, multiply 7 by 5, and subtract the resulting product, 35 from h i. Now, compare the 2 of our divisor 25 with the 5 of our remaining dividend 50 on columns h i. Two goes into 5 twice. Set 2 on column f. This changes the 7 already existing on f to a 9. This new quotient figure of 2 is multiplied by the 2 of 25 and the resulting product 4 is subtracted from the 5 on h. This 2 multiplied by the 5 of 25 gives the product of 10, which is subtracted from the 10 on h i. This clears the abacus of its dividend, leaving the correct quotient 9 on column f.

DISCUSSION

The 2 of 25 is equal to 2 of 225. So, before making a final decision as to quotient figure placement, we had to compare the second digit of the divisor with the second digit of the dividend. Five of 25 is greater than the second 2 of

225. So, this tells us that our first quotient figure must be set on the column immediately to the left of the dividend. In this example, it was column f. There will then be no separation between quotient figure and dividend. Column f, then, is the marker column from which we count off positions. Our first quotient figure was 7. We multiplied 7 by 2. The resulting product was 14. This was the first multiplication of 7 and, therefore, it was subtracted from the first position. In subtracting the 4 of 14, we, of course, had to use one of the secrets for subtraction. We then multiplied this 7 by the 5 of 25. This was the second multiplication of 7, and the resulting product of 35 was subtracted from the second position. This left 50 as our dividend on h i. In our next comparison, the 2 of 25 is smaller than the 5 of 50. This means that the quotient figure resulting from this comparison is to be separated from this dividend digit by one unused column. Two goes into 5 two times. So, counting to the left from column h, on which the first digit of our dividend is now set, we see that column g will be the separation column, and column f which already contains the quotient figure 7 must be used for the placement of the quotient figure 2. The addition of a new quotient figure 2 to the 7 already existing on f corrects 7 upward, changing it to a 9. Multiplying the new quotient figure 2 by the first digit of our divisor, the 2 of 25, gives us a product of 4. This is the first multiplication of our quotient figure, and this product 4 is subtracted from the first position. But it is subtracted from the second column of the first position, or, in other words, from the 5 on column h. Next, we multiplied the quotient figure 2 by the 5 of 25, giving us a product of 10. This is the second multiplication of our quotient figure, and therefore, the product 10 must be subtracted from the second position. Ten being a two-digit number must be subtracted from both columns of this position, or from columns h i, clearing the board and leaving 9 as the correct quotient on column f.

From the foregoing example, you should be able to appreciate the self-corrective nature of upward correction. For this reason, if you are in doubt as to what quotient figure to use, it will be to your advantage deliberately to choose a quotient figure known to be too low. By so doing, you will take advantage of the benefits of automatic upward correction.

DOWNWARD CORRECTION

The correction downward of a quotient figure found to be too large is another case. Like all calculations performed on the abacus, it is automatic, but it is *not* self-corrective as is upward correction. Here is how downward correction is done. When you set a quotient figure on the abacus, you then multiply it by various digits of the divisor, and subtract the resulting products from the dividend, bearing in mind the rules for positioning. Now, if the quotient figure you have chosen is too large, there will come a time during this process of multiplication and subtraction when you will discover this fact. It may be that your discovery will come after you have multiplied the incorrect quotient figure by one or more digits of the divisor. When this happens, you must first reduce the quotient figure by subtracting from it the difference between the incorrectly high quotient figure and your new trial quotient figure. This number which you subtract is called the *corrective figure*, and most people like to reduce quotient figures by a corrective figure of 1 for each trial. Then multiply this corrective figure by those digits of the divisor which you have already multiplied by the incorrect quotient figure, and for which subtractions have been made. The products obtained by multiplying the corrective figure by these digits of the divisor are returned to the dividend in accordance with the rules of positioning, already described. Once you have done this, continue as before, multiplying the revised quotient figure by the remaining digits of the divisor, and go on as if nothing unusual had happened. An example should serve to make things clear. Let's divide 729 by 27. The correct answer will be 27, but watch what happens. Set the divisor 27 on a b and the dividend 729 on g h i, again i being a unit column. Now, compare the 2 of 27 with the 7 of 729. Two goes into 7 three times. Set the quotient figure 3 on column e. Multiply this quotient figure 3 by the 2 of 27. Three by 2 is 6. So subtract 6 from column g. Next, multiply this quotient figure 3 by 7 of 27. Three by 7 is 21. This 21 is to be subtracted from g and h. But 21 is larger than the 12 on g and h. So it is at this point that we have discovered our quotient figure to be too high. So subtract 1 from the 3 on column e. This changes this 3 to a 2. Our corrective figure is 1. Multiply this corrective figure 1 by the 2 of 27 and set the product 2 on column g. Next multiply the revised quotient figure 2 by the 7 of 27, and substract 14 from g and h. Now compare the 2 of 27 with the 18 on g and h. Two will go into 18 nine times. So set 9 as our quotient figure on column f. Nine, multiplied by the 2 of 27 is 18. So subtract 18 from g h. Nine multiplied by 7 is 63, which must be subtracted from h i. Sixty-three is larger than the 9 remaining on column i. Therefore, we have once again discovered that our quotient figure is too high. Subtract 1 from the 9 on f changing it to an 8. Multiply the corrective figure 1 by the 2 of 27, and set the product 2 on h. Now, multiply the new quotient figure 8 by the 7 of 27. Eight by 7 is 56, and this 56 must be subtracted from h i. Since 56 is larger than 29, occurring on h i, we know our quotient figure is still too high. Once again, subtract 1 from the 8 on column f, changing it to a 7. Multiply this corrective figure 1 by the 2 of 27 and set the product 2 on column h. Now multiply the revised quotient figure 7 by the 7 of 27. Seven by 7 is 49, and this can be subtracted from the 49 on h i, clearing the abacus and leaving the correct quotient 27 on columns e f.

DISCUSSION

Again, we see the importance of quotient figure placement, and of adherence to the rules of positioning. In our first comparison, the divisor figure 2 was smaller than the dividend figure 7 which means one unused column must separate quotient figure from dividend. In the second comparison, the divisor digit 2 was larger than the 1 of 18. So, this means that the quotient figure must be placed immediately to the left of the dividend. In all three cases of multiplication of the corrective figure by those digits of the divisor which had been multiplied by the incorrect quotient figure, the numbers to be returned to the dividend were one-digit numbers. This means that they were returned to the second column of the position to which they belonged.

Now, let's review just a part of that example and make an observation or two. Set up the abacus with 27 on a b, 2 on e, and 189 on g h i. This is the point at which we have already established the correctness of our first quotient figure 2 and are ready to make our second set of comparisons. Compare the 2 of 27 with the 18 of 189. Two into 18 is 9. So set the quotient figure 9 on column f. Now, multiply this quotient figure 9 by the 2 of 27. Nine by 2 is 18. So subtract 18 from g h. Now multiply this 9 by the 7 of the divisor 27. Nine by

7 is 63. But you cannot subtract 63 from 9. So far as the abacus is concerned, then, you have not yet multiplied 9 by 7 because no recording of such multiplication and its impossible subtraction has been made. Now we know that the correct quotient figure for column f is 7. So, instead of reducing the quotient figure 9 by a corrective figure of 1 each time, let's subtract 2 from the 9, and change it to the correct quotient figure 7 at one stroke. We multiply the corrective figure 2 by the 2 of 27. This gives us a product of 4. Our marker column is f, and this is the first multiplication of the corrective figure, so it must be set on the first position. Four is a one-digit number, so it must be set on the second column of this position, in other words, on column h. Now we are ready to multiply the new quotient figure 7 by the 7 of 27. Seven by 7 is 49. We subtract 49 from h i, clearing the abacus and leaving the correct quotient 27 on e f.

Observe, again, the importance of quotient figure placement and of multiplication of the corrective figure by those digits which have been multiplied by the unintentionally high quotient figure, and the return of the resulting product of this corrective figure multiplication to the dividend in accordance with rules of positioning.

THE TREATMENT OF ZEROS

If we are working with a divisor which contains one or more zeros in the middle of the number, these digits are treated just as any other number, that is, when the quotient figure is multiplied by the various digits of the divisor, its multiplication by a 0 results in a product of 0. When this happens, this 0 occupies its position in accordance with the already established rules of positioning. Zero is a one-digit number. So, if we follow the technique described earlier of using the index finger and middle finger of the right hand to mark off positions, the 0, being a one-digit number, will occur under the middle finger in its appropriate position. Working through an entire example will not be necessary to make this clear. Let's just set a quotient figure of 7 on column e, and retain our divisor in memory. Our divisor is 30,006. First, we would multiply the quotient figure 7 by 3, giving us 21. This is the first multiplication of 7, and would be subtracted from the first position, columns f g. Seven by 0 is 0—second multiplication; therefore, subtraction must be from the second position. Placing the index finger on column f and the middle finger on column g, we say "position one". Moving this pair of fingers one place to the right, we say "position two". The 0 is subtracted from the column on which the middle finger rests, in this case column h. The third multiplication of 7 is by the second 0 of the divisor, the product being 0. This is subtracted from the second column of the third position, column i. Again, 7 by 0 is 0—fourth multiplication; therefore, subtraction is from the fourth position. Seven by 6 is 42—fifth multiplication; therefore, the 42 would be subtracted from the fifth position. Now, consider: e is the marker column. So, to find the fourth and fifth positions, we would again place the index finger on f, middle finger on column g—position one. One place to the right is position two; one place to the right, position three; one place to the right, position four. Therefore, the 0 resulting from the fourth multiplication is subtracted from column j. Moving our index finger and middle finger over one place to the right gives us the fifth position. You will remember that the fifth multiplication of 7 was by 6. Seven by 6 is 42. So the 4 of 42 is subtracted from the column on which the index finger rests. Again, this is column j, the same column from which we just subtracted our 0. The 2 of 42 is subtracted from column k, on which your middle finger is resting.

This technique of using the index finger and middle finger to identify positions is all right for a beginner. But, with the coming of skill and speed, you will find yourself automatically traveling along the abacus with no conscious effort of position counting. When this happens, the process of division will begin to be automatic for you. In order to reach this goal of automatic operation of the abacus, you should take advantage of every opportunity for practice. Divide your telephone number by two- or three-digit numbers. An interetsing exercise is to divide 987,654,321 by all of the numbers from 1 to 100, recording your quotient each time. Before recording it, of course, you should multiply it to verify its correctness, and for some additional practice in multiplication. Only through exercises of this type, can you gain the skill which you will need for making use of the abacus in your daily life.

FROM PROCESS TO APPLICATION

If you have been faithful to this point, you should be comforted to know that you have now learned all of the processes of arithmetic as they are performed on the Cranmer Abacus. What follows next is a discussion of some of the ways in which these processes may be put to work for you.

THE EXTRACTION OF ROOTS

There are several methods by which square, cube, fourth, and other roots can be extracted using the abacus. These methods consist of a combination of subtraction and division. A description of these methods would require considerable space and might still leave you unable to extract the specific root you need. So, if you have need to extract a given root of a number, we suggest the following procedure. In a table of logarithms look up the logarithm of the number whose root you want to extract. Then, divide this logarithm by the number of the root. The quotient will be the logarithm of the actual root you are seeking. For example, the square root of 729 is 27. Suppose you do not know this, and want to extract the square root of 729. Look up the logarithm of 729, and divide it by 2. The quotient will be the logarithm of 27, the square root of 729. The sixth root of 729 is 3. Dividing the log of 729 by 6 will produce, as a quotient, the logarithm of 3, etc.

Tables of logarithms are available in Braille from the American Printing House for the Blind, 1839 Frankfort Avenue, Louisville 6, Kentucky.

THE TREATMENT OF FRACTIONS

Common fractions, such as $\frac{1}{2}$, $\frac{3}{5}$, $\frac{5}{8}$, etc., or mixed number fractions, such as 2-$\frac{5}{8}$, 7-$\frac{3}{16}$, cannot be handled by the Cranmer Abacus in their original form. Neither can they be handled by any other calculating machine in general use. However, the arithmetic computation associated with fractions can be handled very nicely on the abacus. You will, of course, need either a second abacus or Braille writing equipment for the purpose of recording your work. One example should be enough to give you the idea. Suppose you want to multiply $\frac{5}{8}$ by $\frac{2}{3}$. Such a problem is written graphically: $\frac{5}{8}$ by $\frac{2}{3}$ = $\frac{5}{8}$ by $\frac{2}{3}$. We are then able to multiply the numerators 5 by 2, record the resulting product 10, and, again, we are able to multiply the denominators 8 by 3, and record our resulting product 24. Representing this graphically, we would have $\frac{5}{8}$ by $\frac{2}{3}$ = $\frac{5}{8}$ by $\frac{2}{3}$ equals $\frac{10}{24}$ equals $\frac{5}{12}$. This example is enough to give you the idea, but its simplicity may lead you to question the value of the use of the abacus with fractions. If you have come to such a conclusion, we leave you to ponder the problem 4-$^{37}/_{64}$ by 33-$^{22}/_{85}$. Within a space of two or three minutes, a skilled operator of the abacus should be able to complete this problem, expressing it either as a mixed-number fraction or as a decimal fraction. Thus, the arithmetic associated with the treatment of common fractions and mixed-number fractions can be done quickly and efficiently with the abacus, when to do so is necessary.

Still another way to treat fractions is to convert them to decimal equivalents. One-half would be expressed as .5; $^1/_4$ as .25; $^3/_8$ as .375. The mixed-number fraction 2-$^3/_8$ becomes 2.375.

CALCULATIONS CONTAINING DECIMALS

In the section on addition, we already mentioned that the addition of decimals is done in a straightforward manner. The unit mark is used as a decimal point, and both addition and subtraction can be handled with no need of further explanation. The multiplication and division of numbers in which decimals occur presents only one problem, and that is the location of the decimal point in the product of quotient.

DIGITS VERSUS DECIMALS

To this point in our work with the abacus, we have centered our attention upon specific columns. We have concerned ourselves with the correct placement of digits *upon specific columns*. The decimal point is *not* a digit, and in working with decimals we must concern ourselves not only with the columns themselves, but with the areas *between* columns. A decimal point is a mark of separation. Digits to the left of the decimal point are considered to be whole numbers. Digits to its right are considered to be fractional numbers. Thus, in the number 2.5, the 2 which is to the left of the decimal is a whole number 2. But, the 5 to the right of the decimal point is .5, or the common fraction $^5/_{10}$. Similarly, .25 is considered to be the common fraction $^{25}/_{100}$. In the number 6.25, 6 is a whole number followed by the decimal point of separation and 25 or the common fraction $^1/_4$. So, 6.25 becomes 6-$^1/_4$. Therefore, in working with decimals, we can never say that the decimal point is *on* column e, or *on* column h, etc. We must say the decimal point is *to the left* of column e, *to the right* of column e, or *between* columns d and e, or *between* columns h and i. Notice that the unit marks on your abacus fall between columns and do not occur on the columns themselves. This should help you remember the importance of this matter. Decimals do not occur *upon* columns, they occur *between* columns.

MULTIPLICATION OF DECIMALS

In our earlier discussion of multiplication, we asked you to set the multiplicand on your abacus in such a way that its last digit occurred on a unit column.

This procedure is important not only in helping you locate the last digit of the product, but also in working calculations involving decimals. Locating the decimal point in the product is simplicity itself, no matter how involved the calculation seems to be, if you will set the last whole-number digit of the multiplicand upon a unit column, and if you follow two simple procedures. We will outline these procedures individually, and follow each unit with an appropriate example. First, when the multiplier is a mixed-decimal fraction, that is, a number containing both whole-number digits and a decimal fraction, the decimal point moves to the right from its original setting in the multiplicand by as many places as there are whole number digits in the multiplier, plus one, because we are working with the process of multiplication. For our example, 384 multiplied by 2.8, the multiplicand is 384. So, this is set on the abacus in such a way that the 4 of 384 occurs on a unit column, that is, a column immediately to the left of one of the unit marks. Let's just call this column g. Next, multiply 384 by 28 in a straightforward manner, forgetting for the moment that the multiplier is 2.8. When we multiply 384 by 28, our product is 10,752. Now, to locate the decimal point in this product, let's place our finger upon the unit mark immediately to the left of which we had originally set the 4 of 384. In other words, place your finger on the unit mark which falls between g and h. This is your starting place, and it represents the decimal point of the multiplicand, because the whole number 384 can be thought to be 384.00. Now, the decimal point in the product is moved to the right of the original setting of the decimal point in the multiplicand by a number of places equal to the number of whole-number digits in the multiplier, plus one more place because this is the process of multiplication. From your starting place between columns g and h, move one place to the right. The multiplier was 2.8. So, it had just one whole-number digit, namely 2. Next, move one more place to the right because we are working with the process of multiplication. The first place to which you moved was between columns h and i, and the second place, the one required by the process of multiplication, is between columns i and j. Therefore, this is where the decimal point occurs, between columns i and j. Looking at our product, we see that the decimal occurs between the 5 on column i and the 2 on column j, making our product 1,075.2.

If, instead of 384, our multiplicand had been 3.84, we would have followed precisely the same procedure, except we would have set the multiplicand in such a way that the 3 of 3.84 occurred on the unit column. We would have then counted two places to the right in precisely the same way. If your problem is entered correctly upon the abacus, location of the decimal point is controlled by the number of whole-number digits in the multiplier, and so far as the multiplicand is concerned, you can set it and forget it — until the time comes to find your starting place.

Suppose now that our multiplier had been .28 instead of 2.8. In such a case, we would have found our starting place, that is, the unit mark between columns g and h, and since there would be no whole-number digits in the multiplier, we would move just one place to the right to allow for the process of multiplication. This would have placed the decimal between columns h and i and given us a product of 107.52.

The second procedure for the multiplication of decimals concerns the situation in which the multiplier is a pure decimal fraction whose first significant figure is separated from the decimal point by one or more 0's. In this situation, the decimal point of the product moves one place to the right of its original setting in the multplicand to allow for the process of multiplication, and then to the left by a number of places equal to the number of 0's which separate the decimal from its first significant digit. Let's use the same example, with 384 as our multiplicand, and .00028 as the multiplier. We would set 384 so that the 4 occurs on a unit column, which we will again call column g. We would then multiply it by 28. This time we would make no allowance for the 0's immediately following the decimal in the multiplier, so far as the position concept is concerned. Just multiply 384 by 28. This would give us a product of 10,752. Now let's find the decimal point. Place your finger on the starting place, the unit mark between columns g and h. Now move one place to the right to allow for the process of multiplication, in other words between columns h and i. Now, since our multiplier is .00028, we have three zeros separating the decimal point from 2 of 28. So, from our place between columns h and i, we move to the left three places: first place, back to the starting place between g and h; second place, between f and g; third place, between columns e and f, and this is where the decimal point occurs. Our decimal point then is immediately to the left of column f, giving us the product of .10752.

Now, suppose that the multiplicand were a decimal fraction having many zeros between decimal point and first significant digit, for example, .000384. We would find a unit mark to use as a starting point, and leave the first three columns immediately to its right in the zero condition, setting the significant digits 384 on the fourth, fifth, and sixth columns to the right of our starting decimal point. We would multiply straightforward, and do our counting from the original starting decimal point, following the rule previously described. So here, too, so far as the multiplication is concerned, if proper entry is made of the multiplicand, you can set it and forget it.

THE DECIMAL IN DIVISION

It is just as easy to locate the decimal point in a quotient as it was to find it in the product of multiplication. In the procedures to be described, the dividend is to the multiplicand what the divisor is to the multiplier, and where we moved to the right in multiplication, we will move to the left in division. The procedure, then, is this: Set the dividend in such a way that its last whole-number digit occurs on a unit column. Then, divide it by the divisor in a straightforward way. To find the decimal point in the quotient, place your finger on your starting point, that is, the original decimal setting of the dividend. Next, move to the left as many places as there are whole number digits in the divisor, plus one additional place because this is the process of division. This will represent the location of the decimal point in the quotient. Let's do an example: 6,255 divided by 4.5. Upon your abacus set the dividend 6,255 in such a way that the last 5 occurs upon a unit column. Let's call this column j. Now divide it by the divisor 45 in a straightforward way. Your quotient will be 139. But, now we have to find the decimal point for the quotient. Place your finger on the starting point, the unit mark between columns j and k. The divisor is 4.5. This means we have one whole-number digit to the left of the decimal in the divisor, plus one additional place because this is the process of the division. The first place to which we moved was between columns i and j; the second place, between columns h and i, and this is where the decimal is. Examining the abacus then, we find our quotient to be 1,390.00.

If, instead of 6,255, our dividend had been 6.255, we would have set it in such a way that the 6 of 6.255 would have fallen upon column j, the unit column. It is important that you enter the dividend correctly, but, having entered it, you need have no further concern with it, because you are working with decimals, than you would if you were working a straightforward problem in whole-number division. Just as in the case of the multiplicand, once you set it, you can forget it.

The second procedure for the division of decimals is concerned with a divisor which is a decimal fraction having one or more zeros between the decimal point and the first significant digit. Suppose, this time, we divide 6,255 by .0045. We enter our dividend just as before, and divide by 45 in a straightforward manner. Then, procedure is as follows: Beginning with our starting point, we move to the left one place for the process of division, and to the right by a number of places equal to the number of zeros between the decimal and the first significant digit of the divisor. So, place your finger on your starting point, the unit mark between j and k. Go one place to the left for the process of division, and, since the divisor is .0045, we have two zeros between decimal and first significant digit. So, we go to the right two places. This places the decimal point between columns k and l. This gives us a quotient of 1,390,000.00.

If our dividend were a decimal fraction having one or more zeros between the decimal point and the first significant digit, we enter it just as if we were working a problem in multiplication, that is, we would select our unit mark, and, if the dividend were .006255, we would use the first two columns to the right of the unit mark to represent the zeros, and follow it by 6255. Division should be in a straightforward way, and location of the decimal point would begin from the starting point, just as we have previously outlined.

THE ABACUS AS A CALENDAR

Because of the bulkiness of Braille calendars, and because of the need to replace them every year, it might be of interest to blind people to know that the abacus makes an excellent calendar, that is, it can be used to determine the day of the week on which a given date will occur at any time in the future. It can also be used to determine the day upon which a date which has already passed has occurred, for example, your birthday, or wedding anniversary, or some date important to you or to a friend. Here's how it's done.

Suppose this is Thursday, the first of March, and you want to know on what day November 15 will fall. We will call November 15 the target date; November, the target month; March 1, the present date, and March, the present month. On the abacus you enter the total number of days in the present month. From this you subtract the number of the present date, and to this figure, you add a series of numbers equal to the number of days in the intervening months. These are the months separating the present month from the target month. When you reach the target month, enter a number equal to the target date. This will give you the total number of days between present date and target date. Divide this number by 7, and this will not only give you the number of intervening weeks, but, if there is a remainder, it will represent the difference between the present day and the target day. Let's work through one problem to give you the idea. This is Thursday, March 1, and our target date is November 15. March is the present month. So, on your abacus, enter 31 for March. Subtract from this the present date, which is 1. This leaves 30. Now, to this 30 add the number of days for each of the intervening months. For April, we add 30; for May, 31; for June, 30; for July, 31; for August, 31; for September, 30; for October, 31 and, for November, our target month, we add 15, the number of our target date. Our abacus should now show a total of 259. This means that there are 259 days from March 1 to November 15. Now, divide this 259 by 7. As it happens, the number divides evenly and our quotient is 37, with no remainder. Since there is no remainder, it means that our present day and our target day are the same day of the week, namely Thursday. If our remainder had been 5, we would count forward 5 days from Thursday: Friday, 1; Saturday, 2; Sunday, 3; Monday, 4; Tuesday, 5; and in such a case our target date would have fallen on a target day of Tuesday.

If you want to look ahead a number of years, the procedure is the same, except that you do the additional step of multiplying 365 by the number of years just to save yourself some trouble. In such a case, be sure to include an extra day for every leap year crossed by your calculations.

Now, we will outline the procedure for finding a date which has already passed, but we will not work through an example, since, by now, you should understand our terminology, and have gained enough insight from the example just worked to be able to do it on your own. Suppose we want to locate the day on which October 18, 1910 occurred. 1910 would be our target year, October, our target month, and 18, our target date. First, determine the target date for the present year, that is, October 18 in the current year. Then, enter the number of the current year on your abacus, and from this subtract the number of the target year. Multiply the remainder by 365. This will represent the number of non-Leap Year days between the target date in the target year and the target date in the present year, that is between October 18, 1910 and October 18 of the present year. Next, to this figure add the number of Leap Year days. We suggest that you do it in this way. On the appropriate column of your abacus, move up counters one at a time, as you say the various Leap Years: 1912, '16, '20, '24, '28, '32, '36, 1940, etc. When you have entered the last Leap Year day, you will have the total number of days between October 18, 1910, and October 18 of the present year. Divide this by 7, and this will give you the number of weeks. Should there be a remainder, you would count backwards, that is, if October 18 of the present year is on Wednesday, and you have a remainder of 1, you would count back just 1 day to Tuesday. If you had a remainder of 5, and October 18 of the present year happens to be a Saturday, you would count back 5 days from Saturday: Friday, 1; Thursday, 2; Wednesday, 3; Tuesday, 4; Monday 5. In such a case, October 18, 1910 would have fallen on a Monday. In case you would like to check your results, October 18, 1910 was actually a Tuesday.

CONCLUSION

In presenting the foregoing information, our aim has not been to teach you arithmetic. Instead, we have tried to make available to you the means of enjoying the speed, efficiency, convenience, and independence which the Cranmer Abacus affords blind people. If you have read this work thoroughly, practiced faithfully, and understood all of the principles which have been set forth here, we feel sure that the Cranmer Abacus will become your "LITTLE BEADY BUDDY" for life.

PRACTICE EXERCISES

The following examples have been designed to give you practice in the use of the Cranmer Abacus for the Blind. The answers will be found at the end of these exercises. We suggest that you work each problem and record your answers. Then, consult the answers to check yourself. In this way you should be able to minimize memorization of problems and correct answers.

ADDITION

A1.	A2.	A3.	A4.	A5.	A6.	A7.
1	3	9	4	5	7	7
3	9	9	4	5	7	9
6	6	6	4	6	5	5
4	4	1	8	1	9	3
1	7	4	1	9	8	1
2	1	5	5	9	2	6
7	2	7	2	9	3	4
9	4	3	9	9	4	6
3	3	3	3	9	5	4
8	6	3	7	4	8	1
5	9	4	6	2	1	1
7	5	2	6	1	9	1
4	8	2	4	1	6	1
6	6	6	4	6	4	5
1	6	6	1	6	5	9

A8.	A9.	A10.	A11.	A12.	A13.	A14.
9	37	17	38	35	38	88
2	45	46	46	28	17	66
7	81	38	64	46	26	44
3	26	55	79	37	94	99
4	90	90	87	94	96	11
9	74	74	66	44	69	22
6	15	29	34	66	57	55
5	68	33	91	13	18	77
3	16	11	23	91	90	33
2	99	86	11	58	60	
7						
8						
2						
1						
4						

A15.	A16.	A17.	A18.	A19.	A20.
365	284	545	446	159	101
487	849	654	287	268	202
925	729	776	191	374	358
564	545	163	866	464	946
361	630	926	464	885	857

A21.
1234
3875
9458
1865
2223
5001
6006
1991
9119

A22.
1221
2112
3223
2332
4334
3443
5445
4554
6556
5665
7667
6776
8778
7887
9889
8998

A23.
7047
6375
2208
2147
2108

A24.
1961
1927
1929
1956
1925

A25.
7351
2646
3869
1199
6446

A26.
6464
4466
3951
5133
1050

A27.
1492
1620
1664
1776
1789

A28.
123,456,789
987,654,321

A29.
123,456,789
123,456,789
123,456,789
123,456,789
123,456,789
123,456,789
123,456,789
123,456,789
123,456,789

MULTIPLICATION

M-1. 37
3

M-2. 37
9

M-3. 37
15

M-4. 37
21

M-5. 37
27

M-6. 123
281

M-7. 123
458

M-8. 3367
38

M-9. 3367
83

M-10. 3367
281

M-11. 3367
182

M-12. 3367
6

M-13. 3367
12

M-14. 3367
18

M-15. 3367
24

M-16. 1443
7

M-17. 1443
21

M-18. 1443
35

M-19. 1443
49

M-20. 1443
63

M-21. 315
11

M-22. 594
11

M-23. 772
11

M-24. 844
11

M-25. 271
82

M-26. 271
164

M-27. 271
246

M-28. 271
328

M-29. 231
481

M-30. 231
1443

M-31. 231
2405

M-32. 231
3367

M-33. 231
4329

SUBTRACTION

S- 1.	64 46	S-11.	3210 1234
S- 2.	78 29	S-12.	487 289
S- 3.	92 84	S-13.	5882 4887
S- 4.	91 19	S-14.	1962 1492
S- 5.	365 277	S-15.	432 339
S- 6.	406 173	S-16.	4444 3555
S- 7.	678 565	S-17.	9881 8889
S- 8.	784 576	S-18.	188 89
S- 9.	9999 1234	S-19.	7101 789
S-10.	8765 5555	S-20.	987,654,321 123,456,789

DIVISION

D- 1.	6 into 222	D- 9.	3,367 into 794,612
D- 2.	12 into 444	D-10.	3,367 into 461,279
D- 3.	18 into 666	D-11.	3,367 into 10,101
D- 4.	37 into 888	D-12.	9 into 30,303
D- 5.	272 into 33,456	D-13.	15 into 50,505
D- 6.	371 into 45,633	D-14.	3,367 into 70,707
D- 7.	123 into 63,345	D-15.	3,367 into 90,909
D- 8.	11 into 37,037	D-16.	1,443 into 20,202

ANSWERS TO PRACTICE EXERCISES

ADDITION

A- 1.	67	A-10.	479	A-19.	2,150
A- 2.	79	A-11.	539	A-20.	2,464
A- 3.	70	A-12.	512	A-21.	40,772
A- 4.	68	A-13.	565	A-22.	88,880
A- 5.	82	A-14.	495	A-23.	19,885
A- 6.	83	A-15.	2,702	A-24.	9,698
A- 7.	63	A-16.	3,037	A-25.	21,511
A- 8.	72	A-17.	3,064	A-26.	21,064
A- 9.	551	A-18.	2,254	A-27.	8,341

A-28 1,111,111,110 A-29. 1,111,111,101

MULTIPLICATION

M- 1.	111	M-12.	20,202	M-23.	8,492
M- 2.	333	M-13.	40,404	M-24.	9,284
M- 3.	555	M-14.	60,606	M-25.	22,222
M- 4.	777	M-15.	80,808	M-26.	44,444
M- 5.	999	M-16.	10,101	M-27.	66,666
M- 6.	34,563	M-17.	30,303	M-28.	88,888
M- 7.	56,334	M-18.	50,505	M-29.	111,111
M- 8.	127,946	M-19.	70,707	M-30.	333,333
M- 9.	279,461	M-20.	90,909	M-31.	555,555
M-10.	946,127	M-21.	3,465	M-32.	777,777
M-11.	612,794	M-22.	6,534	M-33.	999,999

SUBTRACTION

S- 1.	18	S-11.	1,976
S- 2.	49	S-12.	198
S- 3.	8	S-13.	995
S- 4.	72	S-14.	470
S- 5.	88	S-15.	93
S- 6.	233	S-16.	889
S- 7.	113	S-17.	992
S- 8.	208	S-18.	99
S- 9.	8,765	S-19.	6,312
S-10.	3,210	S-20.	864,197,532

DIVISION

D-1.	37	D-5.	123	D- 9.	236	D-13.	3,367
D-2.	37	D-6.	123	D-10.	137	D-14.	21
D-3.	37	D-7.	515	D-11.	3	D-15.	27
D-4.	24	D-8.	3,367	D-12.	3,367	D-16.	14